Explicit sum-distinct sets below Bohman's constant: the Erdős problem #1 disproof made effective

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Abstract

Erdős asked (1931) whether a set $A \subseteq \{1, \dots, N\}$ whose $2^{|A|}$ subset sums are distinct must satisfy $N \gg 2^{|A|}$. The best construction for twenty-eight years was $N \le 0.22002 \cdot 2^{|A|}$ (Bohman, 1998). On 28 August 2026 a pre-release GPT-6 Astra, run autonomously by Epoch AI, produced a Lean-verified disproof: for every $\varepsilon > 0$ there are such sets with $N \le \varepsilon 2^{|A|}$. The argument is ineffective at exactly one step, the approximation of a rational lattice by a primitive one, and no explicit set below Bohman's constant was known. We make that step effective by replacing the Smith normal form in the formal proof with a Hermite basis and a chain perturbation, and we compute the one quantity the proof never computed, the buffer K that controls the perturbation. The result is a family of explicit dissociated sets with certificates that an independent program checks in exact arithmetic: the smallest has 1,701 elements with $N/2^{|A|} = 0.217967$, and the best has 79,092 elements with $N/2^{|A|} = 0.145269$, that is 33.9% below Bohman's constant. Two further results ride along. The base gadget $T = I + \frac{1}{2}P$ of the construction is not special: $T = I + \alpha P$ works for every $0 < \alpha < 1$ (the cube property has a three-line proof for all tilts, and the strip property is decided exactly per (α, b) by a finite structured search), and the best tilt beats $\frac{1}{2}$ at every dimension budget. And the base weights give the first explicit lower bounds for the constant in Siegel's lemma above Bohman's, $C_{2,197} \ge 3.441897$ in the normalisation of Bloom's exposition.

1 Introduction

Write $f(n) = \inf \max A/2^{n-1}$ over dissociated sets $A \subset \mathbb{Z}_{>0}$ with $|A| = n$ (Bloom's normalisation) and $\rho(A) = \max A/2^{|A|} = f/2$ for a particular set. Powers of two give $\rho = 1/2$; Conway and Guy give $\rho \le 1/4$ for large n; Bohman [1] gives $\rho \le 0.22002$ for all large n. From below, $N \ge \binom{n}{\lfloor n/2 \rfloor}$ [2]. Erdős's question, whether ρ is bounded below by a positive constant, was answered in the negative by a machine [3, 4]: a Lean 4 proof, checked against the Formal Conjectures statement with the three standard axioms, that for every $\varepsilon > 0$ there are arbitrarily large n with $f(n) < \varepsilon$.

Bloom's exposition [5] reads the proof as a statement about lattices and isolates its single non-effective step. The present note is the effective version of that step, and the objects it produces. Everything quantitative in it is interpolated from a ledger of certificates that an independent verifier re-derived; no number is typed.

*Independent researcher. Contact: carlos@carlostoledo.co. This work was produced with substantial AI assistance in a certified-verification setting: constructions and exposition were proposed with the help of large language models, and every quantitative claim is decided by outward-rounded interval arithmetic or exact rational arithmetic, with adversarial controls that must fire for the certificate to be accepted. The certificates, not the models, are the evidence, and they are public and re-runnable. Machine-derived and not peer-reviewed.

2 The lattice, with a tilt

Fix odd $b \geq 3$ and a rational $0 < \alpha < 1$, and let $V_m = \{x \in \mathbb{R}^m : \sum x_i = 0\}$. Let $T_\alpha : \mathbb{R}^b \to \mathbb{R}^b$, $(T_\alpha z)_i = z_i + \alpha z_{i+1}$ (indices mod b), and

$$\Lambda_1 = T_\alpha(\mathbb{Z}^b \cap V_b), \qquad v_1 = T_\alpha(e_0) = (1, 0, \ldots, 0, \alpha), \qquad h_1 = \sum v_1 = 1 + \alpha.$$

Bloom's gadget is $\alpha = \frac{1}{2}$. For $s \geq 1$ define, in $\mathbb{R}^{b^{s+1}}$ written as b blocks of size $d = b^s$,

$$\Lambda_{s+1} = \{(\lambda_0 + \beta_0 v_s, \ldots, \lambda_{b-1} + \beta_{b-1} v_s) : \lambda_j \in \Lambda_s,\ \beta \in \Lambda_1\}, \qquad v_{s+1} = (v_s, 0, \ldots, 0, \alpha v_s).$$

Call a pair (Λ, v), $\Lambda \subset V_d$ of full rank and $v \notin V_d$, *admissible* if $\Lambda \cap (-1, 1)^d = \{0\}$ and $tv \in \Lambda + (-1, 1)^d$ implies $|t| < 1$.

Lemma 1 (cube, all tilts)**.** *For every $0 < \alpha < 1$ and odd b, $T_\alpha(\mathbb{Z}^b) \cap (-1, 1)^b = \{0\}$; in particular $\Lambda_1 \cap (-1, 1)^b = \{0\}$.*

Proof. Let $z \in \mathbb{Z}^b \setminus \{0\}$ with $|z_i + \alpha z_{i+1}| < 1$ for all i (indices mod b), and let $m = \max_i |z_i| \geq 1$ be attained at i. Then $\alpha |z_{i+1}| > |z_i| - 1 = m - 1$, so $|z_{i+1}| > (m-1)/\alpha \geq m - 1$, hence $|z_{i+1}| = m$, with the opposite sign (if $z_i = m$ then $\alpha z_{i+1} < 1 - m \leq 0$). Around the odd cycle this gives $z_i = -z_i$, a contradiction. The same inequality at the maximum gives $(1 - \alpha)m < 1$. □

Lemma 2 (strip, structure)**.** *Let $x_0 \in \mathbb{R}$ and $x_1, \ldots, x_{b-1} \in \mathbb{Z}$, not all zero, satisfy $|x_i + \alpha x_{i+1}| < 1$ for all i (cyclic, $x_b = x_0$), and let $m = \max_{1 \leq i \leq b-1} |x_i|$. Then $m < 1/(1 - \alpha)$ and $|x_{b-1}| = m$.*

Proof. If the largest index attaining m were $i^* \leq b - 2$, the argument of Lemma 1 would give $|x_{i^*+1}| = m$ with $i^* + 1 \leq b - 1$. If $m \geq 1/(1 - \alpha)$, then either some $i \leq b - 2$ attains m, contradicting $(1 - \alpha)m < 1$, or only x_{b-1} does, and then $|x_0| > (m - 1)/\alpha$ and $|x_1| > (|x_0| - 1)/\alpha > (m - 1 - \alpha)/\alpha^2$, which with $|x_1| \leq m$ forces $m(1 - \alpha^2) < 1 + \alpha$. □

The strip property of (Λ_1, v_1), $|\sum_i x_i| < 1$ for every such configuration, is then decided exactly for each (α, b) by a finite search that Lemma 2 makes small: for each $m < 1/(1-\alpha)$ and each sign of x_{b-1}, enumerate the integer chains $x_{b-2}, \ldots, x_1$ backwards under $|x_j| \leq m$ and $|x_j + \alpha x_{j+1}| < 1$, intersect the two constraints on x_0, and test whether the open interval for $t = \sum_i x_i$ leaves $(-1, 1)$. The search agrees with a brute-force decision of both properties for $b \leq 9$ (and is re-run against it for $b \leq 7$ at every build of the repository), and it admits every (α, b) used below. For $\alpha = \frac{1}{2}$ this is Bloom's case analysis, formalised in Lean as `composeMatrix_admissible` [4].

Lemma 3 (Bloom [5])**.** *If (Λ_1, v_1) is admissible then (Λ_s, v_s) is admissible for every $s \geq 1$. In particular $\Lambda_s \cap (-1, 1)^{b^s} = \{0\}$.*

Proof. Let $y \in \Lambda_{s+1} \cap (-1, 1)^{bd}$. Block j of y lies in $(\Lambda_s + \beta_j v_s) \cap (-1, 1)^d$, so $|\beta_j| < 1$ by admissibility of (Λ_s, v_s); hence $\beta \in \Lambda_1 \cap (-1, 1)^b = \{0\}$, hence every block lies in $\Lambda_s \cap (-1, 1)^d = \{0\}$. The strip property for (Λ_{s+1}, v_{s+1}) follows in the same way with $\gamma = \beta + t v_1$ in place of β, since v_{s+1} is $v_1 \otimes v_s$. □

Let $r = d - 1$ and let $B_s \in \mathbb{Q}^{r \times r}$ hold the first r coordinates of a basis of Λ_s as columns; then $\Lambda_s = \mathrm{lift}(B_s)\mathbb{Z}^r$ with $\mathrm{lift}(B) = \binom{B}{-\mathbf{1}^\top B}$, and, with $h = 1 + \alpha$ and $c = |\det T_\alpha| = 1 + \alpha^b$,

$$\Delta_s(\alpha, b) := |\det B_s| = \frac{(1 + \alpha^b)^{(b^s - 1)/(b-1)}}{(1 + \alpha)^s}, \tag{1}$$

which is verified exactly against a determinant for small (b, s) at every build and for every instance below. Bloom shows $f(n) \leq \Delta_s(1 + o(1))$ along a sequence of n; the $o(1)$ is where effectiveness is lost.

3 The effective transfer

Let $D = q^s$ for $\alpha = p/q$, a common denominator of B_s, and $A = DB_s \in \mathbb{Z}^{r\times r}$.

Hermite basis. Let $H \in \mathbb{Z}^{r\times r}$ be upper triangular with positive diagonal and $H\mathbb{Z}^r = A'\mathbb{Z}^r$, where A' is A with its rows reversed. (PARI's `mathnfmod` applied to A' with $|\det A| = D^r\Delta_s$ as modulus takes 19 seconds at $r = 728$ and 328 seconds at $r = 2196$; FLINT's generic routine takes 53 minutes at $r = 728$, and the two agree.) Reversing coordinates $0, \dots, r-1$ and changing basis do not affect Lemma 3, so with $F = H/D$:

$$\|\mathrm{lift}(F)z\|_\infty \geq 1 \qquad \text{for every } z \in \mathbb{Z}^r \setminus \{0\}. \tag{2}$$

Chain perturbation. Let $\mathrm{top}(X)$ append a zero row to X and let $\mathrm{sh} \in \mathbb{Z}^{(r+1)\times r}$ have ones at positions $(i+1, i)$. For an integer $t \geq 1$ set

$$C = \mathrm{top}(tH) - \mathrm{sh}, \qquad S = \begin{pmatrix} I_r & 0 \\ -\mathbf{1}^\mathsf{T} & 1 \end{pmatrix}, \qquad M = SC = t\,\mathrm{lift}(H) - E, \quad E := S\,\mathrm{sh}.$$

Lemma 4 (saturation)**.** *$M\mathbb{Z}^r$ is a saturated sublattice of $\mathbb{Z}^{r+1}$. Consequently, if $a \in \mathbb{Z}^{r+1}$ is primitive with $a^\mathsf{T}M = 0$, then $\{c \in \mathbb{Z}^{r+1} : a \cdot c = 0\} = M\mathbb{Z}^r$.*

Proof. Delete row 0 of C. Entry (i', j) of the remaining $r \times r$ matrix is $tH_{i'+1,j} - [j = i']$; since H is upper triangular this vanishes for $j < i'$ and equals -1 for $j = i'$. The minor is unimodular, so the gcd of the maximal minors of C is 1 and $C\mathbb{Z}^r$ is saturated. $S \in GL_{r+1}(\mathbb{Z})$ maps saturated lattices to saturated lattices. □

The primitive normal is explicit: $w_0 = 1$, $w_{j+1} = t\sum_{i\leq j} w_i H_{ij}$ for $j = 0, \dots, r-1$ gives $w^\mathsf{T}C = 0$, and $a = S^{-\mathsf{T}}w$, i.e. $a_i = w_i + w_r$ for $i < r$ and $a_r = w_r$. Since $a_0 - a_r = w_0 = 1$ the vector is primitive. This is `transferWeights` of the Lean proof with the Smith normal form replaced by the Hermite form; the leading term is $a_r = t^r \det H + \dots$.

The buffer. E is fixed and small: its rows are 0, $e_0^\mathsf{T}, \dots, e_{r-2}^\mathsf{T}$ and $(-1, \dots, -1, +1)$. Put

$$K := D \cdot \|EH^{-1}\|_{\infty\to\infty},$$

so that for every real z, $\|Ez\|_\infty \leq \|EH^{-1}\|\,\|Hz\|_\infty = K\|Fz\|_\infty \leq K\|\mathrm{lift}(F)z\|_\infty$. This is the one quantity the ineffective proof leaves existential (`exists_perturbation_buffer` in the Lean); it is an exact rational, one per lattice.

Lemma 5 (relation gap)**.** *Let $Q = 2^k$ and $t = \lceil (Q + K)/D \rceil$. If $c \in \mathbb{Z}^{r+1}$, $\|c\|_\infty < Q$ and $a \cdot c = 0$, then $c = 0$.*

Proof. By Lemma 4, $c = Mz$ with $z \in \mathbb{Z}^r$. If $z \neq 0$ then by (2) and the buffer inequality $\|c\|_\infty \geq Dt\|\mathrm{lift}(F)z\|_\infty - K\|\mathrm{lift}(F)z\|_\infty \geq Dt - K \geq Q$. □

Lemma 6 (binary blocks; Lean `exists_binary_block_set`)**.** *If $a \in \mathbb{Z}_{>0}^{r+1}$ satisfies the conclusion of Lemma 5 for $Q = 2^k$, then $\mathcal{A} = \{a_i 2^j : 0 \leq i \leq r,\ 0 \leq j < k\}$ has $(r+1)k$ distinct elements and distinct subset sums.*

Proof. Two subsets with equal sums give $\sum_i a_i c_i = 0$ with $c_i = \sum_{j\in S_i} 2^j - \sum_{j\in S_i'} 2^j \in (-2^k, 2^k)$, hence $c = 0$, hence $S_i = S_i'$ by uniqueness of binary expansions. Distinctness of elements is the case of two coefficients $\pm 2^j$. □

Theorem 1. *For $b = 13$, $s = 3$, $\alpha = \frac{11}{20}$ ($d = 2{,}197$, $r = 2{,}196$, $D = 8{,}000$, $\Delta_s = 0.290062$) and $k = 36$ (so $t = 8{,}589{,}941$, $K = 48{,}362.5$), the set $\mathcal{A}$ of Lemma 6 is a dissociated set of $n = 79{,}092$ positive integers, $\mathcal{A} \subseteq \{1, \dots, N\}$ with $N = 2^{k-1} \max_i a_i$ of 23,809 decimal digits, and*

$$\frac{N}{2^n} = 0.145269 < 0.22002, \qquad \textit{equivalently} \qquad f(79092) \le 0.290537 < 0.44004.$$

For $b = 9$, $s = 2$, $\alpha = \frac{3}{5}$ ($d = 81$, $K = 242.5$) and $k = 21$ the same construction gives a dissociated set of $n = 1{,}701$ integers with $N/2^n = 0.217967$, the smallest set below Bohman's constant in this family.

Proof. Lemmas 1–6 with the exact values recorded in the certificates; the strip property of (Λ_1, v_1) for these (α, b) is decided by the structured search. The ratio is $\max_i a_i / 2^{kr+1}$, computed exactly. □

4 Instances and verification

Table 1 lists every instance the independent verifier has passed on the machine that built this document (40 instances from 11 certificates, 33 of them below Bohman's constant); Table 2 lists the lattices with their buffers. For fixed (b, s, α) the ratio decreases to $\Delta_s/2$ as k grows: at finite k it exceeds the limit by at most the factor $(1 + (K + D)/2^k)^r$ from the rounding of t together with the lower-order terms of the recurrence, which are below $2.91e-08\%$ in the instance of Theorem 1. The buffer grows only linearly with the dimension in every instance computed ($K \approx 1.1\,d$ for $s = 2$ and $\alpha = \frac{1}{2}$, $b = 7, \dots, 15$; $K \approx 1.6$–$2.4\,d$ for $s = 3$, $b = 5, \dots, 13$), so $k = \lceil \log_2(rK) \rceil + 6$ already brings the ratio within one percent of $\Delta_s/2$. A tilt with denominator q pays through $D = q^s$: at $d = 2197$ the tilt $\frac{11}{20}$ carries $K \approx 22\,d$ while $\|EH^{-1}\|$ itself is 3.5, so small denominators are the lever. With Bloom's gadget no $s = 2$ instance can beat Bohman ($\Delta_2 \to 4/9 > 0.44004$) and $b = 9$, $s = 3$ ($d = 729$) is the smallest that does; with a tilt the two-level lattices cross at $d = 81$.

Two programs are provided. `build.py` constructs; `verify.py` rebuilds Λ_s from (b, s, α) and checks, independently of the construction's intermediate objects: that $H\mathbb{Z}^r = A'\mathbb{Z}^r$ (exact rational solves, integrality both ways); that the minor of $C = S^{-1}M$ has determinant ± 1 (a FLINT determinant, not the triangular argument); that the FLINT nullspace of M^{T} is one-dimensional with primitive generator a; that the recorded K is at least the recomputed operator norm; that $Dt - K \ge 2^k$; positivity, distinctness, n, N and the ratio. The inputs not recomputed at full size are Lemma 1 and the strip decision, which are elementary, and the iteration Lemma 3, formalised in Lean for $\alpha = \frac{1}{2}$; all are checked exhaustively for $d \le 9$, and at $d = 729$ a falsification probe tested 18,612 lattice vectors (every basis vector, sums and differences of neighbouring pairs, and random small integer combinations) and found none inside the open cube. Everything is exact: no floating point enters any certificate. Every certificate was verified twice, once on the bench that built it and once, from the certificate file alone, on the machine that built this document; only the second run admits an instance into the tables above.

5 The tilt as a lever

Larger tilts pay in covolume and gain in height, so the best tilt depends on the dimension budget, and it beats $\alpha = \frac{1}{2}$ at every budget: under $d \le 100$ the tilt 0.58 at $b = 9$, $s = 2$ gives $\Delta = 0.4313$ in dimension 81, where Bloom's gadget cannot beat Bohman below dimension 729; under $d \le 500$, $\alpha = \frac{3}{4}$, $b = 21$, $s = 2$ gives 0.3440 in dimension 441; under $d \le 1000$, $\alpha = \frac{4}{5}$, $b = 31$, $s = 2$ gives 0.3186 against 0.3539; under $d \le 10^4$, $\alpha = 0.65$, $b = 21$, $s = 3$ gives 0.2351 against 0.2964; under $d \le 10^6$, $\alpha = 0.65$, $b = 31$, $s = 4$ gives 0.1417 against 0.1975 (a float scan; the certificates are exact). Two-parameter cyclic gadgets $I + aP + bP^2$ fail the cube condition from $b = 7$ on, through period-three patterns, so within cyclic gadgets the one-parameter tilt is the lever.

6 A corollary: explicit lower bounds in Siegel's lemma

In the normalisation of Bloom's exposition (after Aliev [6]), let C_d be the least constant such that every nonzero $a \in \mathbb{Z}^d$ admits a nonzero $x \in \mathbb{Z}^d$ with $a \cdot x = 0$ and $\|x\|_\infty^{d-1} \le C_d \|a\|_\infty$. Bombieri and Vaaler give $C_d \ll \sqrt{d}$; the recorded lower bound is Schinzel's $C_d \ge 1$, and Bloom remarks that the Astra construction should yield explicit lower bounds. It does, with no further work: the base weights $a \in \mathbb{Z}^d$ of Lemma 5 are 2^k-relation-free, so every admissible x has $\|x\|_\infty \ge 2^k$, whence $2^{k(d-1)} \le C_d \max a$, that is

$$C_d \;\ge\; \frac{2^{kr}}{\max_i a_i} \;=\; \frac{1}{f},$$

with f the Bloom-normalised ratio of the instance, an exact rational, printed truncated in Table 3. A Bohman-type set gives $C_d \ge 1/0.44004 = 2.2725$ for all large d; these are the first explicit values above that, and they grow with the instance as the construction predicts.

7 Limits and what would come next

Bohman's bound is for all large n; Theorem 1 is for specific n. An effective asymptotic statement $f(n) \le n^{-c/\log\log n}$, the form Bloom expects, would follow from a structural bound on K as a function of (b, s), because k must only exceed $\log_2(rK)$ by a constant. The buffer is measured here and not proved: the Hermite basis is computed, not analytic, and an analytic triangular basis of Λ_s with an explicit inverse would settle it. Bigger tilted instances ($\alpha = 0.65$, $b = 21$, $s = 3$ at $d = 9261$) need a Hermite path beyond PARI at that size. Nothing here concerns the lower bound $N \ge \binom{n}{\lfloor n/2 \rfloor}$ or the true order of f.

Reproduction

Every constant in this paper is interpolated from a certificate record in the `cert-machine` repository, so the document cannot quote a number the machine did not certify. The certificates (`certs/erdos1/`), the ledger they enter through (`certs/erdos1-ledger.json`), the instrument (`instruments/erdos1/`: `lattice.py`, `build.py`, `verify.py`, `gadget.py`, `siegel.py`) and the detached verifier (`tools/verify_erdos1.py`, needs python-flint) are in the repository; `python3 tools/verify_erdos1.py certs/erdos1/cert-b9-s2-a3_5-k21.json` re-derives the smallest set in seconds. The macros of this document are written by `instruments/erdos1/certnumbers.py`

from the ledger. Sources pinned in `corpus/sources/erdos1/`: the problem page and Bloom's exposition (2026-09-15 fetch), the forum thread, the Lean resolution (`tadamcz/erdos1` at commit `0e395153`), Epoch's paper extract.

References

[1] T. Bohman, A construction for sets of integers with distinct subset sums, Electron. J. Combin. 5 (1998), R3.

[2] Q. Dubroff, J. Fox, M. W. Xu, A note on the Erdős distinct subset sums problem, SIAM J. Discrete Math. 35 (2021).

[3] T. Adamczewski, T. F. Bloom, FrontierMath Erdős, Epoch AI, 1 September 2026.

[4] GPT-6 Astra (OpenAI), Lean disproof of Erdős problem #1, packaged by T. Adamczewski, `https://github.com/tadamcz/erdos1`, 2026.

[5] T. F. Bloom, exposition of the disproof, `https://www.erdosproblems.com/1`, 3 September 2026.

[6] I. Aliev, Siegel's lemma and sum-distinct sets, Discrete Comput. Geom. 39 (2008), 59–66.

b	s	α	d	k	$n = \lvert\mathcal{A}\rvert$	digits of N	$N/2^n$	$f = N/2^{n-1}$	vs 0.22002
9	2	$\frac{3}{5}$	81	16	1,296	390	0.297911	0.595822	above
9	2	$\frac{3}{5}$	81	18	1,458	439	0.233466	0.466932	above
9	2	$\frac{3}{5}$	81	20	1,620	488	0.220057	0.440114	above
9	2	$\frac{3}{5}$	81	21	1,701	512	0.217967	0.435934	below
9	2	$\frac{3}{5}$	81	22	1,782	536	0.216929	0.433858	below
9	2	$\frac{3}{5}$	81	24	1,944	585	0.216180	0.432359	below
9	2	$\frac{3}{5}$	81	28	2,268	683	0.215928	0.431857	below
11	2	$\frac{13}{20}$	121	20	2,420	728	0.251217	0.502434	above
11	2	$\frac{13}{20}$	121	22	2,662	801	0.214039	0.428079	below
11	2	$\frac{13}{20}$	121	24	2,904	874	0.206812	0.413624	below
11	2	$\frac{13}{20}$	121	28	3,388	1,020	0.204055	0.408110	below
11	2	$\frac{3}{5}$	121	18	2,178	656	0.228941	0.457881	above
11	2	$\frac{3}{5}$	121	20	2,420	728	0.211918	0.423837	below
11	2	$\frac{3}{5}$	121	22	2,662	801	0.207916	0.415832	below
11	2	$\frac{3}{5}$	121	24	2,904	874	0.206954	0.413908	below
11	2	$\frac{3}{5}$	121	28	3,388	1,020	0.206643	0.413285	below
13	2	$\frac{3}{5}$	169	20	3,380	1,017	0.204743	0.409487	below
13	2	$\frac{3}{5}$	169	22	3,718	1,119	0.196145	0.392289	below
13	2	$\frac{3}{5}$	169	24	4,056	1,221	0.194069	0.388138	below
13	2	$\frac{3}{5}$	169	28	4,732	1,424	0.193433	0.386865	below
21	2	$\frac{3}{5}$	441	20	8,820	2,655	0.294264	0.588529	above
21	2	$\frac{3}{5}$	441	22	9,702	2,920	0.196745	0.393491	below
21	2	$\frac{3}{5}$	441	24	10,584	3,186	0.177898	0.355796	below
21	2	$\frac{3}{5}$	441	26	11,466	3,451	0.173475	0.346950	below
21	2	$\frac{3}{5}$	441	28	12,348	3,717	0.172386	0.344773	below
9	3	$\frac{1}{2}$	729	24	17,496	5,267	0.189259	0.378518	below
9	3	$\frac{1}{2}$	729	26	18,954	5,705	0.179938	0.359875	below
9	3	$\frac{1}{2}$	729	28	20,412	6,144	0.177680	0.355360	below
9	3	$\frac{1}{2}$	729	32	23,328	7,022	0.176980	0.353960	below
31	2	$\frac{4}{5}$	961	22	21,142	6,364	0.352433	0.704865	above
31	2	$\frac{4}{5}$	961	24	23,064	6,943	0.194426	0.388853	below
31	2	$\frac{4}{5}$	961	26	24,986	7,521	0.167372	0.334745	below
31	2	$\frac{4}{5}$	961	28	26,908	8,100	0.161276	0.322552	below
31	2	$\frac{4}{5}$	961	30	28,830	8,678	0.159783	0.319567	below
11	3	$\frac{1}{2}$	1,331	28	37,268	11,218	0.160447	0.320894	below
11	3	$\frac{1}{2}$	1,331	30	39,930	12,020	0.158673	0.317346	below
11	3	$\frac{1}{2}$	1,331	32	42,592	12,821	0.158232	0.316465	below
11	3	$\frac{1}{2}$	1,331	34	45,254	13,623	0.158122	0.316245	below
13	3	$\frac{1}{2}$	2,197	36	79,092	23,809	0.151520	0.303039	below
13	3	$\frac{11}{20}$	2,197	36	79,092	23,809	0.145269	0.290537	below

Table 1: Every verified instance, ordered by dimension and tilt. Ratios are exact rationals rounded to six places.

b	s	α	d	D	Δ_s	K	K/d
11	2	$\frac{13}{20}$	121	400	0.407792	1,660.8	13.73
11	2	$\frac{2}{3}$	121	9	0.413245	217.7	1.80
11	3	$\frac{1}{2}$	1,331	8	0.316172	2,984.1	2.24
13	2	$\frac{2}{3}$	169	9	0.386780	349.9	2.07
13	3	$\frac{11}{20}$	2,197	8,000	0.290062	48,362.5	22.01
13	3	$\frac{1}{2}$	2,197	8	0.302989	5,158.9	2.35
21	2	$\frac{3}{4}$	441	16	0.344050	1,277.1	2.90
31	2	$\frac{4}{5}$	961	25	0.318575	3,460.8	3.60
9	2	$\frac{3}{5}$	81	25	0.431825	242.5	2.99
9	3	$\frac{1}{2}$	729	8	0.353867	1,546.6	2.12

Table 2: The lattices: covolume constant and the computed buffer.

d	α	k	f	$C_d \geq$
81	$\frac{3}{5}$	28	0.431857	2.315583
121	$\frac{13}{20}$	28	0.408110	2.450317
169	$\frac{2}{3}$	28	0.386865	2.584880
441	$\frac{3}{4}$	28	0.344773	2.900463
729	$\frac{1}{2}$	32	0.353960	2.825177
961	$\frac{4}{5}$	30	0.319567	3.129239
1,331	$\frac{1}{2}$	34	0.316245	3.162105
2,197	$\frac{11}{20}$	36	0.290537	3.441897

Table 3: The best verified Siegel bound per dimension.